A Toy Model of Boundary States with Spurious Topological Entanglement Entropy

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Topological entanglement entropy has been extensively used as an indicator of topologically ordered phases. We study conditions for two-dimensional topologically trivial states to exhibit spurious contributions which suffer topological entanglement entropy. We show that if the state at the boundary of a subregion is a stabilizer state, then it has non-zero spurious contribution on the region if, and only if, the state is in a non-trivial one-dimensional $G_1 \times G_2$ symmetry-protected topological (SPT) phase. However, we provide a candidate of a boundary state that has a non-zero spurious contribution but does not belong to any of such SPT phases.

Introduction.– Topologically ordered phases are gapped quantum phases which cannot be detected by conventional local order parameters. Topological entanglement entropy (TEE) [1, 2] has been widely used as an indicator of such topological ordered phases. For ground states in gapped two-dimensional (2D) models, the entanglement entropy $S(A) := -\mathrm{Tr}\rho_A \log_2 \rho_A$ of a region A is expected to behave as

$$S(A) = \alpha|\partial A| - \gamma + o(1)\,, \tag{1}$$

where α is a constant, ∂A is the boundary length and $o(1)$ comprises terms vanishing in the limit of $|\partial A| \to \infty$. TEE is defined as the universal constant term γ [1]. The term γ is shown to be the logarithm of the total quantum dimension of the abstract anyon model under various conditions [1, 3–5].

To extract TEE from a ground state, one can calculate suitable linear combinations of entropies for certain subsystems (e.g., Fig. 1a), known as conditional mutual information (CMI) in quantum information theory, so that the first leading terms cancel out [1, 2].

However, it has been pointed out that Eq. (1) in general could contain an additional term, and thus the above argument does not always work. This additional contribution, called spurious TEE [6, 7], causes positive CMI for states in the trivial phase.

So far, the spurious TEE is seems to be connected to the existence of a 1D symmetry-protected topological (SPT) phase at the boundary of a certain region [6–11]. Spurious TEE seems to be fragile against general local perturbations or small deformation of the regions, but the conditions in which the spurious TEE appears have not yet been fully understood.

A natural question is whether a SPT phase at the boundary is also a *necessary* condition for spurious TEE. In this letter, we study the underlying mechanism of spurious TEE in the trivial phase. We model the degrees of freedom at the boundaries of regions (Fig. 1b) by using Matrix Product States (MPS) [12]. Here we particulary focus on a renormalization fixed-point of the MPS in which the CMI is constant for all the length scales. We then characterize the fixed-points in terms of the operator-algebra quantum error correction [13–15], and derive a formula to calculate the value of the spurious TEE from algebras associated to the single tensor.

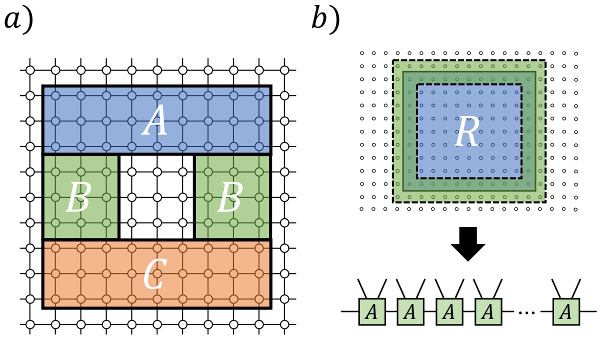

FIG. 1: *a*) A tripartition of a subsystem in a 2D spin lattice to calculate TEE. *b*) For ground states in the trivial phase, the entanglement entropy of region R is determined by a MPS located at the boundary of R (green region). Each tensor A has two physical legs associated to R and its complement respectively.

By using our characterization, we show that if the boundary MPS is a stabilizer state [16], a non-zero spurious TEE implies the MPS is in a non-trivial $G_1 \times G_2$ SPT phase under on-site symmetry actions. In contrast, we also provide a numerical evidence indicating that in general there are boundary states which have non-zero spurious TEE but does not belong to any such SPT phase. To the best of our knowledge, this is the first example of the mechanism of spurious TEE beyond on-site $G_1 \times G_2$ SPT phase at the boundary.

MPS model of boundary states.– We consider a translation-invariant ground state $|\psi\rangle$ defined on a 2D spin lattice with size N. When a ground state is in the trivial phase, it can be (approximately) constructed from a product state only by a constant-depth local unitary circuit [17, 18]. More precisely, there exists a set of unitaries $\{V_i\}$ such that

$$|\psi\rangle = V_w V_{w-1} \dots V_1 |0\rangle^{\otimes N}\,,$$

where the depth $w = \mathcal{O}(1)$ is a constant of N and each V_i is a product of local unitaries acting on disjoint sets of neighboring spins within radius $r = \mathcal{O}(1)$.

Let us divide the lattice into a connected region R and its complement R^c. Entanglement between R and R^c is invariant under local unitaries $U_R U_{R^c}$ and therefore we can undo some parts of the circuit. Hence, $S(R)_\rho$ is equivalent to that of a tensor product of an entangled state $|\phi\rangle_{RR^c}$ around the boundary ∂R and $|0\rangle$s at the rest part. We call $|\phi\rangle_{RR^c}$ as *the boundary state* of R (Fig. 1b).

A constant-depth circuit can increase the Schmidt-rank by at most a constant. Therefore $|\phi\rangle_{RR^c}$ is written as a Matrix Product State (MPS) (Fig. 1b):

$$|\phi\rangle_{RR^c} = \sum \mathrm{Tr}(A^{i_1 j_1} \dots A^{i_l j_l}) |i_1 \dots i_l\rangle_R |j_1 \dots j_l\rangle_{R^c} \,,$$

where $A^{i_k j_k}$ is a $D \times D$ matrix with a constant bond dimension $D = \mathcal{O}(1)$ (here, we assume that all the tensors are the same due to the translation-invariance [28]). Each local basis $\{|i_k\rangle\}$ corresponds to a coarse-grained site which consists several neighboring spins so that the correlation length of the MPS is exactly zero. We denote by $\mathcal{H}$ and $\mathcal{K}$ the Hilbert spaces associated to $|i\rangle_R$ and $|j\rangle_{R^c}$. In this notation, there is an isometry $V : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathcal{H} \otimes \mathcal{K}$, $V^\dagger V = I$ such that the MPS has the form [19, 20]

$$|\phi\rangle_{RR^c} = V^{\otimes l} |\lambda_D\rangle^{\otimes l} \,, \tag{2}$$

where $|\lambda_D\rangle = \sum_{k=1}^{D} \sqrt{\lambda_k} |kk\rangle$ is an entangled state with the Schmidt rank D. V acts on two separated sites of neighboring $|\lambda\rangle$s. In the following, we especially consider the case where $|\lambda_D\rangle$ is the maximally entangled state $|\omega_D\rangle := \sum_{i=1}^{D} \frac{1}{\sqrt{D}} |ii\rangle$ for the simplicity. We expect we do not lose much generality by this reduction, although we leave an extension for future works.

When R is an annulus like ABC in Fig. (1)a, we obtain two boundary states at the inner and outer boundaries. The ground state has a spurious TEE for R if one of these boundary states have a non-trivial CMI

$$I(A:C|B)_\rho := S(AB) + S(BC) - S(B) - S(ABC) > 0$$

for a tripartition $R = ABC$ such that B separates A from C. Importantly, the value of CMI matches to that of the tri-information [21], which also used to extract TEE [1], for the class of states we are considering.

Due to the monotonicity of CMI, non-zero value of the spurious TEE implies that the CMI of an open boundary MPS must be positive as well. We formalize a family of such open boundary MPS $\{\phi^{(n)}\}_{n\geq 0}$ with different length n defined as

$$\phi^{(0)} := |\omega_D\rangle\langle\omega_D|_{A_1 A_2} \,,$$
$$\phi^{(n)} := \mathcal{V}_{A_{2n}A_{2n+1} \to B_n E_n} \left(\phi^{(n-1)} \otimes |\omega_D\rangle\langle\omega_D|_{A_{2n+1}A_{2n+2}} \right) ,$$

where $\mathcal{V}_{A_{2n}A_{2n+1}}(X) = VXV^\dagger$ is the isometry map (Fig. 2). For the convenience we relabel A_{2n+2} by C_n so that each $\phi^{(n)}$ is a state on $A_1 \otimes (B_1 \otimes E_1) \otimes (B_2 \otimes E_2) \otimes \cdots \otimes (B_n \otimes E_n) \otimes C_n$. A_1 and C_n represents the unfixed boundary condition.

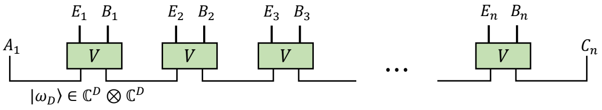

FIG. 2: A schematic picture of the family of states $\phi^{(n)}$. $|\omega_D\rangle = \sum_i \frac{1}{\sqrt{D}} |ii\rangle$ is the D-dimensional maximally entangled state and V is an isometry from $\mathbb{C}^D \otimes \mathbb{C}^D$ to $\mathcal{H} \otimes \mathcal{K}$.

After tracing out $R^c = E_1 ... E_n$, we have a family of mixed states $\{\rho^{(n)}\}_{n\geq 0}$ defined by

$$\rho^{(0)} := |\omega_D\rangle\langle\omega_D|_{A_1 A_2} \,,$$
$$\rho^{(n)} := \mathcal{E}_{A_{2n}A_{2n+1} \to B_n} \left(\rho^{(n-1)} \otimes |\omega_D\rangle\langle\omega_D|_{A_{2n+1}A_{2n+2}} \right) ,$$

where $\mathcal{E} = \mathrm{Tr}_E \circ \mathcal{V}$ is a completely-positive and trace-preserving (CPTP) map. We denote CPTP-map $\mathcal{E}(\cdot \otimes |\omega_D\rangle\langle\omega_D|)$ by $\tilde{\mathcal{E}}$. We then have

$$\rho^{(n+1)} = \tilde{\mathcal{E}}_{C_n \to B_{n+1} C_{n+1}} (\rho^{(n)}) \,. \tag{3}$$

The whole family is obtained by iteratively applying $\tilde{\mathcal{E}}$:

$$\rho^{(n)} = \tilde{\mathcal{E}}_{C_n \to B_{n+1} C_{n+1}} \circ \cdots \circ \tilde{\mathcal{E}}_{A_2 \to B_1 C_1} (\rho^{(0)}) \tag{4}$$

(recall that $C_0 = A_2$). We will simply denote the concatenated map in Eq. (4) by $\tilde{\mathcal{E}}^{(n)}$. When we trace out R instead of R^c, we obtain the complement chain which we will denote by $\{\sigma^{(n)}\}$. We also define $\mathcal{F} := \mathrm{Tr}_B \circ \mathcal{V}$ and $\tilde{\mathcal{F}}(\cdot) = \mathcal{E}^c(\cdot \otimes |\omega_D\rangle\langle\omega_D|)$.

$\{\phi^{(n)}\}$ has a spurious TEE if $I(A_1 : C_n | B_1 ... B_n)_{\rho^{(n)}}$ is bounded from below by a positive constant. Although $\rho^{(n)}$ has zero correlation length, we might still have a non-trivial length scale for the CMI [22]. We further remove such lengths scale by requiring saturation of the CMI:

$$I(A_1 : C_1 | B_1)_{\rho^{(1)}} = I(A_1 : C_n | B_1 \dots B_n)_{\rho^{(n)}} \,, \forall n. \tag{5}$$

Note that the LHS is always larger for any CPTP-map $\mathcal{E}$. In the rest of the paper we will simply denote $I(A_1 : C_n | B_1 B_2 \dots B_n)_{\rho^{(n)}}$ by $I(A_1 : C_n | B_1 B_2 \dots B_n)_{(n)}$.

While the definition (5) depends on n, it is equivalent to two independent conditions independent of n.

Proposition 1. *Eq.* (5) *is equivalent to*

$$I(A_1 : B_1 C_1)_{(1)} = I(A_1 : B_1 B_2 C_2)_{(2)} \,, \tag{6}$$
$$I(A_1 : B_1)_{(1)} = I(A_1 : B_1 B_2)_{(2)} \,. \tag{7}$$

Moreover, Eq. (7) *is equivalent to*

$$I(A_1 : E_1 C_1)_{(1)} = I(A_1 : E_1 E_2 C_2)_{(2)} \,. \tag{8}$$

Therefore it is sufficient to consider up to $n = 2$.

Characterization by operator-algebra QEC.– We use the theory of operator-algebra quantum error correction (OAQEC) [13–15] to characterize $\{\phi^{(n)}\}$. OAQEC is a general framework of quantum error correction including standard quantum error correction codes [23] and subsystem codes [24]. It allows us to describe what kind of observables is correctable against a given error. For a given CPTP-map $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ representing a "noise", one can always specify the *correctable algebra* $\mathcal{A}_{\mathcal{E}} \subset \mathcal{B}(\mathcal{H})$ that is a C^*-algebra containing all observables whose information is preserved under $\mathcal{E}$ (see Supplymental Material (SM) for more details).

In the following analysis the correctable algebras of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ play a crucial role. We first show that the saturation of the conditional mutual information (5) implies the saturation of these correctable algebras.

Proposition 2. *If Eq.* (5) *holds for* $\mathcal{E}$*, then*

$$\mathcal{A}_{\tilde{\mathcal{E}}} = \mathcal{A}_{\tilde{\mathcal{E}}^{(n)}}\,, \tag{9}$$

$$\mathcal{A}_{\tilde{\mathcal{F}}} = \mathcal{A}_{\tilde{\mathcal{F}}^{(n)}}\,, \tag{10}$$

$$\mathcal{A}_{\mathrm{Tr}_C\circ\tilde{\mathcal{E}}} = \mathcal{A}_{\mathrm{Tr}_C\circ\tilde{\mathcal{E}}^{(n)}}\,, \tag{11}$$

$$\mathcal{A}_{\mathrm{Tr}_C\circ\tilde{\mathcal{F}}} = \mathcal{A}_{\mathrm{Tr}_C\circ\tilde{\mathcal{F}}^{(n)}}\,, \forall n\,. \tag{12}$$

This proposition means that the algebra $\mathcal{A}_{\tilde{\mathcal{E}}}$ represents the information of the input which is faithfully encoded in the output on $B_1 ... B_n C_n$ for all n. In the same way, $\mathcal{A}_{\mathrm{Tr}_C\circ\tilde{\mathcal{E}}}$ represents the perfectly recoverable information encoded on $B_1 ... B_n$.

In general, there are operators carrying "unpreserved" information which are disturbed and cannot be recovered perfectly. Such operators also could contribute to CMI, but it may cause decrease of CMI with respect to n. Prop. 2 does not prevent to have such unpreserved operators and therefore the conditions (9)-(12) are not sufficient for Eq. (5). In fact, we can always assume these conditions by coarse-graining a finite number of channels [29].

Unpreserved information could be split into the local part and the non local part. The local part is outputted in B_1 and does not affect to CMI. The non-local part is outputted in $B_1 C_1$, and therefore it is further disturbed by applying $\tilde{\mathcal{E}}$ on C_1, which induces the decay of CMI. We would like to disregard this unpreserved information, and thus we utilize the concept of complementary recovery property [25] to neglect such information.

Definition 3. *We say a CPTP-map* $\mathcal{E}$ *satisfy complementary recovery property if*

$$\mathcal{A}_{\mathcal{E}^c} = \mathcal{A}'_{\mathcal{E}}\,, \tag{13}$$

where $\mathcal{A}'_{\mathcal{E}}$ *is the commutant of* $\mathcal{A}_{\mathcal{E}}$*.*

Any CPTP-map satisfies $\mathcal{A}_{\mathcal{E}^c} \subset \mathcal{A}'_{\mathcal{E}}$, i.e. any operator recoverable from the output of the complementary channel $\mathcal{E}^c$ should commute with the correctable algebra of the original channel $\mathcal{E}$ (see also SM). The complementary recovery property says the converse of this statement is also true. This property can be characterized by a projection map onto the correctable algebra.

Proposition 4. $\mathcal{E}$ *satisfies the complementary recovery property if, and only if,*

$$\mathcal{E}\left(\mathcal{P}_{\mathcal{A}_{\mathcal{E}}}(\rho)\right) = \mathcal{E}(\rho)\,, \forall\rho \tag{14}$$

or

$$\mathcal{P}_{\mathcal{A}_{\mathcal{E}}} \circ \mathcal{E}^\dagger(O) = \mathcal{E}^\dagger(O)\,, \forall O\,, \tag{15}$$

where $\mathcal{P}_{\mathcal{A}_{\mathcal{E}}} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}_{\mathcal{E}}$ *is the idempotent projecting map onto* $\mathcal{A}_{\mathcal{E}}$*.*

Therefore the complementary recovery property restricts the support of the input to the correctable algebra.

Definition 5. *We say isometry* V *or CPTP-map* $\mathcal{E}$ *satisfies dual complementarity if* $\tilde{\mathcal{E}}$ *and* $\tilde{\mathcal{F}}$ *both satisfy the complementary recovery property.*

Dual complementarity reduces the four algebras in Prop. 2 to two algebras $\mathcal{A} := \mathcal{A}_{\tilde{\mathcal{E}}} = (\mathcal{A}_{Tr_C\circ\tilde{\mathcal{F}}})'$ and $\mathcal{B} := \mathcal{A}_{\tilde{\mathcal{F}}} = (\mathcal{A}_{Tr_C\circ\tilde{\mathcal{E}}})'$. In the following we only consider states satisfying this property. We show that the dual complementarity implies the saturation of the CMI. Furthermore, its value is determined by $\mathcal{A}$ and $\mathcal{B}$.

Theorem 6. *If* V *satisfies dual complementarity, then Eq.* (5) *holds. Let* $\mathcal{A} = \bigoplus_k M_{n_k}(\mathbb{C}) \otimes I_{n'_k}$ *and* $\mathcal{B} = \bigoplus_l M_{m_l}(\mathbb{C}) \otimes I_{m'_l}$*. Then, the value of the CMI is given by*

$$I(A_1 : C_1|B_1)_{(1)} = \sum_k p_k \log\frac{n_k}{n'_k} + \sum_l q_l \log\frac{m_l}{m'_l}\,, \tag{16}$$

where $p_k = \frac{n_k n'_k}{D}$ *and* $q_l = \frac{m_l m'_l}{D}$*. Therefore,* $I(A_1 : C_1|B_1)_{(1)} > 0$ *if and only if*

$$\mathcal{B}' \subsetneq \mathcal{A}\,. \tag{17}$$

Eq. (17) intuitively means there is an operator perfectly encoded in BC whose information cannot be read out by just looking B. Such a non-local information causes the spurious contribution to CMI.

Note that dual complementarity is not a necessary condition for Eq. 5. For example, one can consider $\mathcal{E} = \mathcal{E}'_{A_2} \otimes \mathrm{id}_{A_3}$ such that $\mathcal{E}'$ is not the completely depolarizing channel but $\mathcal{A}_{\mathcal{E}'} = \mathbb{C}I$. The corresponding family satisfies $I(A : C|B) = 0$ for any length, but the map does not satisfy dual complementarity. This is because all the unpreserved information is transfered to B and not C, and thus it cancels out in $I(A : C|B) = I(A : BC) - I(A : B)$.

Relation to SPT phases.– For any $O \in \mathcal{A}$, we always find a corresponding *logical operator* $\tilde{O}_{BC}$ such that

$$\mathcal{V}_{A_2\to BEC}(O_{A_2}|\psi\rangle_{A_2}) = (\tilde{O}_{BC} \otimes I_E)\mathcal{V}_{A_2\to BEC}|\psi\rangle_{A_2}\,,$$

for any $|\psi\rangle \in \mathbb{C}^D$, where

$$\mathcal{V}_{A_2 \to BEC}|\psi\rangle_{A_2} := V^{\otimes n}\left(|\psi\rangle_{A_2} \otimes \bigotimes_i |\omega_D\rangle_{A_{2i+1}A_{2i+2}}\right).$$

The logical operator is not unique in general. The set of all logical operators $\mathcal{L}_\mathcal{A}$ of operators in $\mathcal{A}$ is given as the pre-image of $\tilde{\mathcal{E}}^{(n)\dagger}$:

$$\mathcal{L}_\mathcal{A} := \{O_{BC} \,|\tilde{\mathcal{E}}^{(n)\dagger}(O_{BC}) \in \mathcal{A}\}.$$

$\tilde{\mathcal{E}}^{(n)\dagger}$ is a normal $*$-homomorphism from the pre-image to $\mathcal{A}$ [15]. By the first isomorphism theorem for algebra, the image of the homomorphism is isomorphic to the pre-image upto the kernel.

$$\mathcal{L}_\mathcal{A}/Ker\tilde{\mathcal{E}}^{(n)\dagger} \cong \mathcal{A}.$$

We denote the equivalence class of the logical operators of $O \in \mathcal{A}$ by $\mathcal{L}(O)$.

Suppose the boundary state is in a non-trivial SPT phase under a symmetry of group $G_1 \times G_2$ acting on each tensor as $U(g_1, g_2) = U(g_1)_B \otimes U'(g_2)_E$. The action induces a projective representation $V(g) \otimes V(g)^\dagger$ on the virtual degrees of freedom [26] (see also SM). For instance, it holds that

$$U(g)_{B_1E_1}|\phi^{(1)}\rangle_{A_1B_1E_1C_1} = V(g)^T_{A_1} \otimes V(g)^\dagger_{C_1}|\phi^{(1)}\rangle_{A_1B_1E_1C_1}$$

for $n = 1$. This correspondence reads that $V(g_1) \in \mathcal{A}$ and $V(g_2) \in \mathcal{B}$. $V(g)$ has a logical unitary operator

$$U(g) \otimes U(g) \otimes \cdots \otimes U(g) \otimes V(g) \in \mathcal{L}(V(g))$$

whose support is BC (EC) if $g = (g_1, e)$ ($g = (e, g_2)$). Suppose the state is in a non-trivial SPT phase in the sense that $[V(g_1), V(g_2)] \neq 0$ for some g_1, g_2 [6]. This implies $\mathcal{B}' \subsetneq \mathcal{A}$. Therefore, we can reconfirm that non-trivial $G_1 \times G_2$ SPT phase implies non-zero CMI (under dual complementarity).

The converse direction is entirely non-trivial. The existence of tensor product logical unitaries $U_B \otimes U_C$ is necessary for $\phi^{(n)}$ to being a state in such a SPT phase, but not sufficient as we will see in later. A particular class of V in which the converse also holds is the isometry consists of Clifford gates, i.e. when the MPS is a stabilizer states [16].

Theorem 7. *Let V be an isometry composed of Clifford gates and ancillas $|0\rangle^{\otimes k}$. Then,*

$$I(A_1 : C_n|B_n)_{(n)} > 0\,,\ \forall n \tag{18}$$

if and only if there exists finite groups G_1 and G_2 such that the MPS generated by V is in a non-trivial $G_1 \times G_2$ SPT phase.

The proof is given in SM. Theorem 7 can be applied for all 2D topologically trivial stabilizer states including the 2D cluster state [7]. However, the conclusion is not necessarily true outside of stabilizer states. In fact, one can find a family of boundary states such that all the non-identity logical unitaries cannot be written as $U_B \otimes U_C$.

A non-trivial example.– Let V_U be an isometry that is the Stinespring dilation of $\mathcal{E}_U(\sigma) = \frac{1}{4}\sum_{i=0}^{3}(P_i \otimes P_iU)\sigma(P_i \otimes U^\dagger P_i)$, where P_i ($i = 0, 1, 2, 3$) are the Pauli Matrices ($P_0 = I$). The correctable algebras are $\mathcal{A} = \mathcal{B} = M_2(\mathbb{C})$, therefore boundary states automatically satisfy dual complementarity. The CMI attains the maximum value $I(A : C|B)_{(n)} = 2$. Note that this model is in a $D_2 \times D_2$ SPT phase if $U = I$.

For $n = 1$, each Pauli operator P_i has an unique logical operator $(P_iP_i)_B \otimes U^T(P_i)_CU^*$ [30]. If U is not a Clifford unitary nor diagonal in X or Z-basis, both $U^TX_CU^*$ and $U^TZ_CU^*$ are non-Pauli matrices. This induces non-tensor product logical operators on B_2C_2, which are also unlikely to be a tensor product for $n > 2$ (Fig. 3). By coarse-graining $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}^{(2)}$, we obtain a model with no logical operator form like $U_B \otimes U_C$ but with $I(A : C|B)_{(n)} = 2$.

We can construct a non-trivial 2D translation-invariant model by considering a layer of many copies of this 1D example along the vertical and the horizontal direction (decoupled stacks) as in the case of a 2D weak subsystem SPT phase [27]. The resulting 2D state has a spurious TEE for an arbitrary large dumbbell-like region [7].

One may expect that for the periodic boundary condition the CMI could vanish in such a non-trivial example. Although we do not have any analytical result on that, we numerically sampled U from the Haar measure and then calculate the CMI for closed chains. Fig. 4 suggests that CMI remains to be a positive constant even for the closed boundary, while the value decreases from 2.

Future directions.– A crucial open question is how to characterize/classify the non-trivial example with spurious TEE. Although it should not be in a SPT phase under on-site $G_1 \times G_2$ symmetry, it could be in a SPT phase under other type of symmetry.

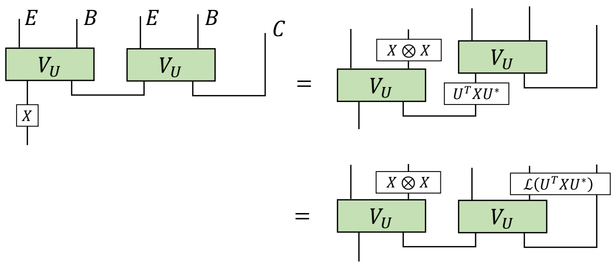

FIG. 3: A logical operator of Pauli X in the example. This is a unique logical operator of X supported on BC. For general U, the logical operator is no longer a tensor product of unitaries on B and C for $n = 2$. We expect this to be hold for $n > 2$ as well.

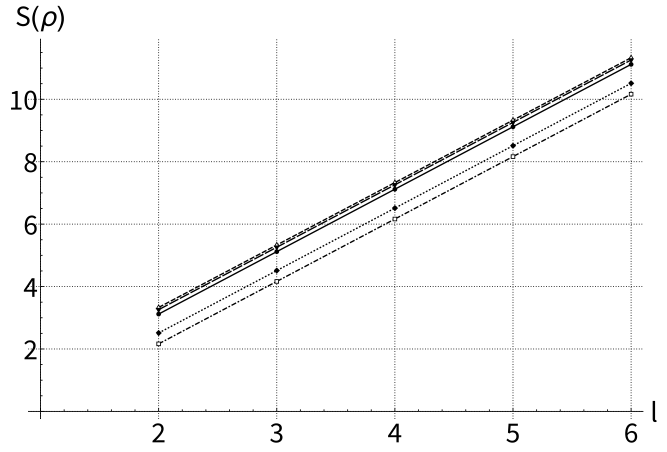

FIG. 4: The numerical result on the entropy $S(\rho_{ABC})$ of the example with closed boundary for 5 samples of U from the Haar measure. l is the length of the spin chain ($l = 6$ is 12-qubit). From the data, $S(\rho) = 2l - c_0$ with a constant $c_o > 0$ up to 10^{-7} error. Since any reduced state of the example is the completely mixed state, it shows that $I(A : C|B)_{(n)} = c_0$ for any tripartition ABC such that B separates A from C.

Generalization to more broad class of boundary MPS is desired. One possible extension is considering boundary states without dual complementarity. Dual complementarity neglect all information outside of $\mathcal{A}$, but in general one has some "noisy" information localized on B (or E). It may be possible to extend the correctable algebra by adding operators carrying such information. Another important direction is considering general injective MPS including (2). We expect that general injective MPS can be decomposed into protected and unprotected parts as in Ref. [11] such that the effect of the unprotected part vanishes exponentially as the conditioning system grows. We leave these problems for future works.

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[28] This assumption is slightly stronger than the translation-invariance, since generally tensors can depend on the direction of the edge and it can even contain some "corner" tensors. However, such corner contributions cancel out in the calculation of CMI.

[29] For any $\mathcal{E}$ we have $\mathcal{B}(\mathbb{C}^D) \supset \mathcal{A}_{\tilde{\mathcal{E}}} \supset \mathcal{A}_{\tilde{\mathcal{E}}^{(2)}} \supset \ldots \supset \mathbb{C}I$, and thus we cannot have an infinitely long sequence of strictly different C^*-algebras. Therefore there is $m \in \mathbb{N}$ such that the conditions hold by redefining $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}^{(m)}$ ($\mathcal{F} \equiv \tilde{\mathcal{F}}^{(m)}$).

[30] The uniqueness follows from the fact that there is no stabilizer supported on BC.